Review of Logarithms

Many physiological processes are dependent on the difference of two quantities. For example, the rate of diffusion of a substance across a cell membrane is dependent on the concentration difference of the substance outside and inside the cell (e.g., $C_{out} - C_{in}$). Still other processes are dependent on the ratio of two quantities. An example of this is the electrical potential (voltage) across a cell membrane, which is dependent on the ratio of concentrations of permeant ions on the two sides of the membrane (e.g., C_{out}/C_{in}). In the former case, the solution of problems usually only involves simple algebraic manipulations (addition, subtraction, multiplication and division); in the latter case, however, the solution of the problems often involves the use of logarithms.

In this write-up, we will review the use of logarithms, and the mathematical properties of the logarithm and antilogarithm (power) functions. In the future, we will assume that you are reasonably competent with simple algebra, as well as the use of these functions. If you find yourselves uncomfortable with these topics, do not panic! Spend an evening or two reviewing them, or seek help immediately from one of the instructors or teaching assistants.

Our discussion will be limited to the use of common or base-10 logarithms. Another commonly used type of logarithm is the natural or base-*e* logarithm. Do not confuse the two different types of logarithms; the use of natural logarithms will result in incorrect results when solving problems in this course (see below regarding use of calculators).

The use of a calculator to compute logarithms (and antilogarithms) is permitted during quizzes and examinations. Nevertheless, we strongly urge you to become familiar with the properties of logarithms (discussed below). In many cases, simple observation will allow you to deduce the correct answer to a numerical problem without the need of a calculator. In addition, understanding the properties will help you identify errors resulting from incorrect use of your calculator.

Working with logarithms

The logarithm of a number is simply the exponent that indicates the power to which ten must be raised to produce that number. That is, if

$$y = 10^x$$

then x is the logarithm of y, and is written

$$x = \text{Log}\ y.$$

It is therefore simple to identify the logarithm of integral powers of ten. For example

$y = 1000$	means $y = 10^3$;	therefore, Log 1000	=	3.
$y = 100$	means $y = 10^2$;	therefore, Log 100	=	2.
$y = 10$	means $y = 10^1$;	therefore, Log 10	=	1.
$y = 1$	means $y = 10^0$;	therefore, Log 1	=	0.
$y = 0.1$	means $y = 10^{-1}$;	therefore, Log 0.1	=	–1.
$y = 0.01$	means $y = 10^{-2}$;	therefore, Log 0.01	=	–2.
$y = 0.001$	means $y = 10^{-3}$;	therefore, Log 0.001	=	–3.

The above table illustrates several properties of logarithms. Numbers greater than 1 have positive logarithms; numbers less than one (but greater than zero) have negative logarithms. Logarithms do not exist (are undefined) for zero or negative numbers, because you cannot raise 10 to any power such that the result will be zero or a negative number.

The above table shows how to determine logarithms of numbers that are integral powers of ten, but what about all other numbers? For example, consider Log 300. Clearly, since 300 is greater than 100 but less than 1000, Log 300 will be some number between 2 and 3. Similarly, consider Log 0.003. Since 0.003 is greater than 0.001 but less than 0.01, then Log 0.003 will be a number between –3 and –2. This type of observation will be sufficient for you to select the correct answer for many exam questions. However, if a more accurate answer is needed, you can use your calculator (or a table); you would find that Log 300 = 2.48 and Log 0.003 = –2.52.

Logarithms of products and ratios

The logarithm of a product can be split into two simpler Log calculations, since

$$\text{Log } xy = \text{Log } x + \text{Log } y$$

Log tables are only published for values ranging between 1 and 10, because the above relationship then allows you easily to compute logarithms for any number. First, you write the number in scientific notation, for example:

$$\text{Log } 0.003 = \text{Log } (3\times10^{-3})$$

Then you simply evaluate each Log separately:

$$\text{Log } (3\times10^{-3}) = \text{Log } 3 + \text{Log } 10^{-3} = 0.48 + (-3) = 0.48 - 3 = -2.52$$

The logarithm of a ratio can also be split into two simpler Log calculations:

$$\text{Log } x/y = \text{Log } x - \text{Log } y$$

and from this, you should be able to show that

$$\text{Log } x/y = -\text{Log } y/x$$

In the types of problems alluded to earlier involving computation of logarithms of a ratio of two quantities, it really does not matter how you set up the ratio (i.e., whether you use x/y or y/x). The values of the logarithms will differ only in sign. This does not mean that the sign is unimportant! Our point here is that you can readily determine whether the final numerical result is positive or negative by brief consideration of the physical problem.

Given the properties described above (i.e., logarithms of products and ratios), you will find that it is quite easy to compute logarithms of many numbers (without using a table or calculator) if you simply memorize one simple fact, namely Log 2 = 0.30. In essence, this means that every time a number doubles, its logarithm increases by 0.30; every time a number is halved, its logarithm decreases by 0.30. A few examples will illustrate this:

Log 4	=	Log (2×2)	=	0.30 + 0.30	=	0.60
Log 8	=	Log (4×2)	=	0.60 + 0.30	=	0.90
Log 16	=	Log (8×2)	=	0.90 + 0.30	=	1.20
Log 5	=	Log (10/2)	=	1.00 – 0.30	=	0.70
Log 2.5	=	Log (5/2)	=	0.70 – 0.30	=	0.40
Log 1.25	=	Log (2.5/2)	=	0.40 – 0.30	=	0.10

Antilogarithms (or inverse logarithms)

The antilogarithm is simply the inverse computation. Namely, if you know the logarithm of a number, you can compute the value of the number itself by taking the antilogarithm. The antilogarithm is defined as

$$y = \text{antilog } x = 10^x$$

As an example, suppose you want to compute the antilogarithm of 3.5. Clearly, since 3.5 is a number between 3 and 4, the antilogarithm will result in some number between 1000 and 10,000. Again, this observation will be sufficient for many problems in this course. If more accuracy is required, then you could use your calculator (or a Log table). The antilog of 3.5 equals 3160.

Since Log and antilog are inverse functions, then this means that

$$10^{\text{Log } x} = x, \quad \text{and} \quad \text{Log } 10^x = x .$$

Hence, you can use this information to modify a formula to solve for different desired quantities. For example, consider the following formula:

$$z = \text{Log } x/y .$$

Suppose you are given z and x and are asked to compute y. Simple algebraic rules follow (i.e., if you do something to one side of the equation, you must do the same thing to the other side of the equation):

$$z = \text{Log } x/y$$

$$10^z = 10^{\text{Log } x/y} = x/y$$

From this, y can be calculated directly (i.e., $y = x/10^z$).

Finally, you should also be familiar with some other properties of the antilog function, namely

$$10^{-x} = \frac{1}{10^x} \text{ and } 10^{x+y} = 10^x \times 10^y .$$

Formula-plugging versus understanding the formula

You will find that formula plugging and blind use of calculators may not result in success when solving a problem. A simple example will illustrate this point. During the first few weeks of the course, we will spend time discussing cell membrane potentials (voltages). Suppose we state that the voltage across a certain cell membrane is given by

$$V = (-60\,\text{mV}) \times \text{Log}\left(\frac{[\text{K}^+]_{\text{in}}}{[\text{K}^+]_{\text{out}}}\right)$$

where $[K^+]_{in}$ and $[K^+]_{out}$ are the intracellular and extracellular potassium-ion concentrations, respectively. We might ask a question like:

> The intracellular and extracellular potassium ion concentrations are 140 mM and 4 mM, respectively. What is the membrane potential?

or,

> The membrane potential is –60 mV and the extracellular potassium ion concentration is 4 mM. What is the intracellular potassium ion concentration?

Both of these questions could be solved either directly (first question) or after some simple algebraic manipulations (second question) by simply plugging the appropriate values into the formula. We might, however, ask a different type of question. For example:

> The membrane potential initially is –120 mV. If the extracellular potassium concentration is increased ten-fold, what will the new membrane potential become?

Notice that we do not give you explicit values for either the intracellular or extracellular concentrations. From the equation, however, you should see that the ratio of intracellular to extracellular potassium concentrations is 100-to-1 (i.e., Log 100 = 2, hence V = –120 mV). Increasing the extracellular potassium concentration ten-fold would reduce this ratio by a factor of 10, to 10-to-1. Since Log 10 = 1, this means that the membrane potential would become –60 mV.

Logarithms and units

As a general rule, logarithm or antilogarithm functions operate on numbers that are unitless. For most problems in this course, you will be asked to evaluate logarithms of ratios of two numbers, e.g., Log x/y. x and y alone will have units, but the computation of the ratio results in a unitless value (i.e., the units factor out). If in your calculations you find that the units do not cancel, then you probably have made an error.

There is one notable exception to this rule! It is common to express concentrations of some substances as logarithmic quantities. The values are usually denoted by the prefix "p". Three quantities that you will be concerned with are the hydrogen ion concentration $[H^+]$, the dissociation constant of a weak acid K_a, and the intracellular concentration of calcium $[Ca^{++}]$, which are often represented as

pH	=	$-\text{Log}\,[H^+]$	($[H^+]$ in moles/liter or molar)
pK_a	=	$-\text{Log}\,K_a$	(K_a in moles/liter or molar)
pCa	=	$-\text{Log}\,[Ca^{++}]$	($[Ca^{++}]$ in moles/liter or molar)

The units of these quantities are understood by the above definitions, and need not be reported with the number. For example, if one states that the pH of a solution is 7.3, then it is assumed that the person means that the $[H^+] = 5\times10^{-8}$ moles/liter (or 50 nM). Note, however, that you must not omit the units when referring to an actual concentration.

Common problems with calculators

Most scientific calculators implement both common (base-10) and natural (base-*e*) logarithms (see above). Usually, "log" (or "LOG") is used to denote common logarithm, and "ln" (or "LN") is used to denote natural logarithm. Calculators typically use either "10^x" or "inv log" to denote common antilogarithm. We have seen, however, that some calculator manufacturers use a different scheme to distinguish the different logarithms. Namely, upper-case "Log" is used for common logarithms, and lower-case "log" is used for natural logarithms (this is a convention used by mathematicians).

We have also observed some students using the "10^x" key in an attempt to enter numbers expressed in scientific notation (e.g., 6.02×10^{23}) into their scientific calculators. This often leads to incorrect results. The correct way to enter these values is to use the "enter exponent" key (usually denoted "EEX" or "Exp").

It is your responsibility to familiarize yourself with the correct operation of your calculator! A good self test would be to use your calculator to solve the numerical study problems accompanying this write-up. If you experience problems, then immediately consult a teaching assistant or instructor for help.

Study problems

Table of Common Logs

	x =1	2	3	4	5	6	7	8	9	10
Log	x =0.00	0.30	0.48	0.60	0.70	0.78	0.85	0.90	0.95	1.00

1. Using the above table only, compute the logarithms of the following numbers (check your results with a calculator):

   4000, 6×10^{23}, 2×10^{-15}, 0, 9000, 0.00006, –30

2. Again, without using a calculator, compute the inverse logarithms of the following numbers (check your results with a calculator):

$1.3, \ -2.3, \ 0, \ 0.3, \ 20, \ -7.15$

Hint: express the Log value as the sum or difference of an integer (positive or negative) and a positive decimal fraction between 0 and 1. Then, use the above table and the definition of the power of a sum of two numbers to compute the final result. Example:

$$10^{-8.1} = 10^{0.9-9} = 8\times10^{-9}$$

3. If the pH of a solution changes from 7.0 to 8.0, then by what factor does the $[H^+]$ change (increase or decrease)?

4. A weak acid HA is added to water and partially dissociates to form H^+ (hydrogen ions) and A^-. The following formula relates the concentrations of the different species to the dissociation constant (K_a) of the acid at equilibrium:

$$K_a = [H^+][A^-]/[HA]$$

Derive a formula that would allow you to compute pH given the pK_a, $[A^-]$, and [HA]. Note that the resulting formula will form the basis of important topics related to renal (kidney) and pulmonary (lung) physiology. (Hint: start by taking logs of both sides of the above equation, then regroup the terms using the definition of the Log of a product given above.)

5. The correct answer to problem 4 is: $pH = pK_a + \text{Log}([A^-]/[HA])$. If the pH of the resulting solution is 7.1, and you are told that the pK_a of the acid is 6.1, then what is the ratio of $[HA]/[A^-]$? Can you compute the ratio of $[HA]/[H^+]$? If so, what is it?

Answers to study problems

1. $\text{Log } 4000 = \text{Log } 4\times10^3 = \text{Log } 4 + \text{Log } 10^3 = 0.60 + 3 = 3.60$
$\text{Log } 6\times10^{23} = \text{Log } 6 + \text{Log } 10^{23} = 0.78 + 23 = 23.78$
$\text{Log } 2\times10^{-15} = \text{Log } 2 + \text{Log } 10^{-15} = 0.30 + (-15) = -14.7$
Log 0 is undefined
$\text{Log } 9000 = \text{Log } 9\times10^3 = \text{Log } 9 + \text{Log } 10^3 = 0.95 + 3 = 3.95$
$\text{Log } 0.00006 = \text{Log } 6\times10^{-5} = \text{Log } 6 + \text{Log } 10^{-5} = 0.78 + (-5) = -4.22$
Log –30 is undefined

2. $10^{1.3} = 10^{0.3+1} = 10^{0.3}\times10^1 = 2.0\times10 = 20$
$10^{-2.3} = 10^{0.7-3} = 10^{0.7}\times10^{-3} = 5.0\times0.001 = 0.005$
$10^0 = 1$
$10^{0.3} = 2$
10^{20} (or 1×10^{20})
$10^{-7.15} = 10^{0.85-8} = 10^{0.85}\times10^{-8} = 7\times10^{-8}$

3. $[H^+]$ decreases by ten-fold: at pH 7, $[H^+] = 10^{-7}$ moles/liter (or 100 nM); at pH 8, $[H^+] = 10^{-8}$ moles/liter (or 10 nM); $10^{-7}/10^{-8} = 10$.

4. $K_a = [H^+][A^-]/[HA]$

$\text{Log } K_a = \text{Log } ([H^+][A^-]/[HA])$	
$\text{Log } K_a = \text{Log } [H^+] + \text{Log } ([A^-]/[HA])$	(definition of Log of a product)
$-\text{Log } K_a = -\text{Log } [H^+] - \text{Log } ([A^-]/[HA])$	(multiply both sides by -1)
$pK_a = pH - \text{Log } ([A^-]/[HA])$	(definition of pK_a and pH)
$pH = pK_a + \text{Log } ([A^-]/[HA])$	(add Log term to both sides)

5. $7.1 = 6.1 + \text{Log } ([A^-]/[HA])$

$1.0 = \text{Log } ([A^-]/[HA])$	(subtract 6.1 from both sides)
$-1.0 = \text{Log } ([HA]/[A^-])$	(Log $x/y = -$Log y/x, see above)
$[HA]/[A^-] = 10^{-1.0} = 0.1$	(take antilogarithm)

You can compute $[H^+]$ and $[HA]/[A^-]$, but you do not have enough information to compute either $[HA]$ or $[A^-]$. Hence, you cannot compute $[HA]/[H^+]$.